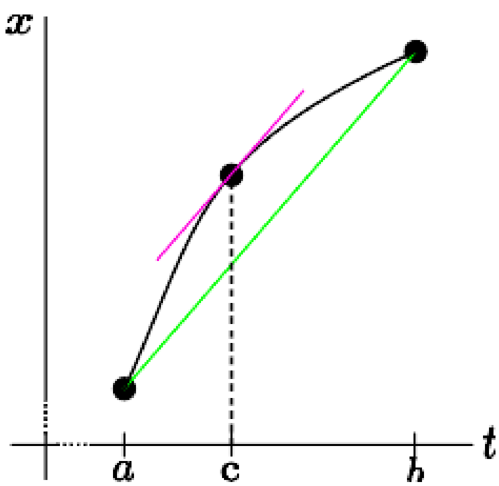
The Mean Value Theorem (MVT)

If $x(t)$ is continuous on $a \leq t \leq b$, and differentiable on $a < t < b$, that is, $x'(t)$ is defined for all t, $a < t < b$, then

$$\frac{x(b) - x(a)}{b - a} = x'(c) \qquad \text{for some } c, \text{ with } a < c < b.$$

Equivalently, in geometric terms, there is at least one point c, with $a < c < b$, at which the tangent line is parallel to the secant line through $(a, x(a))$ and $(b, x(b))$:



Upper and Lower Bounds

We have introduced the notion of upper and lower bounds.

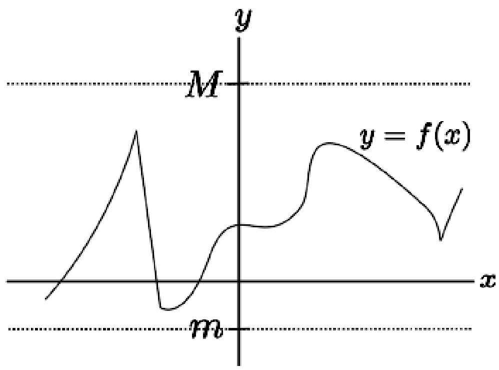
A number M is an **upper bound** of a function $f(x)$ if

$$f(x) \leq M \qquad \text{for all } x$$

and a number m is a **lower bound** of a function $f(x)$ if

$$m \leq f(x) \qquad \text{for all } x$$

We can consider upper and lower bounds on the entire real number line, or on an interval.



$$m \leq f(x) \leq M$$

In other words, an upper bound of a function is a number that is larger than or equal to all values of the function. A lower bound of a function is a number which is smaller than or equal to all values of the function.

Old news

We have been relying on the following fundamental facts whenever we try to understand a function using its derivative. But in fact, these facts are consequences of the mean value theorem.

- If $x'(t) \geq 0$ for all t in (A, B), then $x(t)$ is **increasing or staying the same** over $[A, B]$.
- If $x'(t) > 0$ for all t in (A, B), then $x(t)$ is **strictly increasing** over $[A, B]$.
- If $x'(t) \leq 0$ for all t in (A, B), then $x(t)$ is **decreasing or staying the same** over $[A, B]$.
- If $x'(t) < 0$ for all t in (A, B), then $x(t)$ is **strictly decreasing** over $[A, B]$.
- If $x'(t) = 0$ for all t in (A, B), then $x(t)$ is **constant** over $[A, B]$.

These facts need proofs and their proofs are based on the MVT. The subtlety is that the MVT relates the infinitesimal behavior of the function, the derivative, which is defined at a point, to the macroscopic behavior of the function, the total change over an interval.

Bounding the average rate of change

The equality in the MVT can be used to restrict the range of possible values of the average rate of change and the total change.

More precisely, if there are numbers m and M such that

$$m \leq x'(c) \leq M \quad \text{for all } c \text{ with } a < c < b, \tag{1}$$

(2)

that is, m is a lower bound and M is an upper bounds on $x'(c)$ over (a, b), then the MVT implies the following.

$$m \leq \frac{x(b) - x(a)}{b - a} \leq M \quad \text{(Bounds on the average rate of change)} \tag{3}$$

$$m \cdot (b - a) \leq x(b) - x(a) \leq M \cdot (b - a) \quad \text{(Bounds on the total change)} \tag{4}$$

In other words, a lower bound on the derivative is also a lower bound on the average rate of change, and an upper bound on the derivative is also an upper bound on the average rate of change. Also, the product of a lower bound on the derivative with the length of an interval, is a lower bound on the total change of the function over that interval. Similarly, the product of an upper bound on the derivative with the length of an interval, is an upper bound on the total change of the function over that interval.

When we know the maximum and minimum values of $x'(t)$, we can use them as bounds on $x'(t)$ and obtain the following.

$$\min_{a \leq t \leq b} x'(t) \leq \frac{x(b) - x(a)}{b - a} \leq \max_{a \leq t \leq b} x'(t). \quad \text{(Bounds on the average rate of change)} \tag{5}$$

$$\min_{a \leq t \leq b} x'(t) \cdot (b - a) \leq x(b) - x(a) \leq \max_{a \leq t \leq b} x'(t) \cdot (b - a) \quad \text{(Bounds on the total change)} \tag{6}$$

(7)

In other words, the average rate of change must be in between the maximum and the minimum of the derivative, and the total change must be in between the maximum and minimum of the derivative multiplied by the length of the interval.