Differential notation

Let $y = F(x)$, the **differential of y** is defined as

$$dy = F'(x)dx.$$

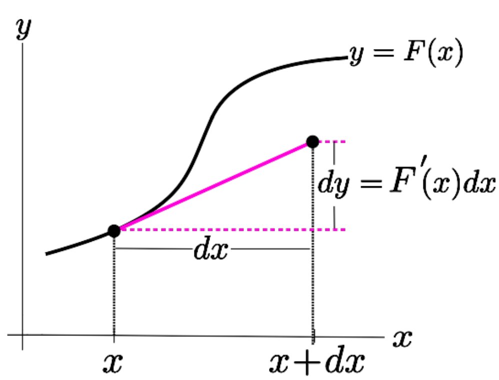
This is also called the **differential of F** and denoted dF.

Rearranging this equation, we get the the Leibniz notation for the derivative, which says the derivative is the ratio of the two differentials dy and dx.

$$F'(x) = \frac{dy}{dx} \qquad \left(\text{or } \frac{dF}{dx}\right)$$

We may think of the differential of x, dx, as a "little bit" of x, and the differential of y, dy, as a "little bit" of y. Here, what we mean by a "little bit" is really an infinitely small bit, we call these infinitely small quantities "infinitesimals." The point is that even though both dy and dx are infinitely small, their ratio is NOT. Their ratio is the derivative $F'(x)$. In other words, the differential notation says that dy is proportional to dx with constant of proportionality $F'(x)$ even though both dy and dx are infinitely small. We use the differential notation as a tool to keep track of how much y changes when x changes by a tiny tiny tiny ... tiny bit.

The geometric picture for the differential is the same as that for linear approximation.



Let us compare the differential notation with the formula for linear approximations.

$$\text{Linear approximation at } x\Delta F \quad \approx F'(x)\,\Delta x, \Delta x \text{ is a finite change in x.}$$
$$\text{Differential notation: } dF \quad = F'(x)\,dx, dx \text{ is a tiny tiny } \dots \text{tiny bit of } x$$

Antiderivatives

An antiderivative of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x).$$

Indefinite integral

Given a function $f(x)$, the **indefinite integral** or **the antiderivative** of $f(x)$ is denoted $\int f(x)dx$. It is the family of functions

$$\int f(x)\,dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$, and C is any constant.

We call $\int$ the **integral sign**, $f(x)$ the **integrand**, and C the **constant of integration**.

The constant of integration is present in this definition because the derivative of a function determines only the shape of the function, but the derivative does not change if the function is shifted up or down by the same constant everywhere.

Uniqueness of the indefinite integral

The indefinite integral

$$\int f(x)\,dx = F(x) + C$$

is termed "indefinite" since it contains an undetermined constant C and is not just one function but a family of infinitely many functions, parameterized by C.

On the other hand, the constant is the only ambiguity of the indefinite integral due to the MVT, which guarantees that any two antiderivatives of the same function can differ only by a constant.

Integrals of powers

$$\int x^p \, dx = \begin{cases} \dfrac{x^{p+1}}{p+1} & \text{if } p \neq -1 \\ \ln(|x|) & \text{if } p = -1. \end{cases}$$

First rules of Integration

	Integration Rules	Differentiation Rules
Constant multiple:	$\int k\, f(x)\, dx = k \int f(x)\, dx$	$d(kF) = k\, dF$
Sum:	$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$	$d(F+G) = dF + dG$

On the other hand, the following naive product and quotient rules do NOT work.

$$\int f \cdot g dx \textbf{DOES NOT EQUAL} \qquad \left(\int f dx\right) \cdot \left(\int g dx\right)$$

$$\int \frac{f}{g}\, dx \textbf{DOES NOT EQUAL} \qquad \frac{\int f\, dx}{\int g\, dx}.$$

Method of substitution

The method of substitution is the integration analogue of the chain rule.

If

$$g(x)\, dx = f(u(x))\, u'(x)\, dx,$$

that is, the differential $g(x)\, dx$ is the result of a chain rule, then

$$\begin{aligned} \int g(x)\, dx &= \int f(u(x))\, u'(x)\, dx \\ &= \int f(u)\, du \\ &= F(u(x)) + C \end{aligned}$$

where $F(u)$ is an antiderivative of $f(u)$.